Hwk for 11/29 can be turned in Wed morning to my office (TUC 358).

Also, Test Corrections for Test #2 can be turned in (last time) on Wed morning to TUC 358.

Test #3 corrections due Wed 12/7!

Using FTC (Fundamental Theorem of Calculus to calculate integrals quickly.)

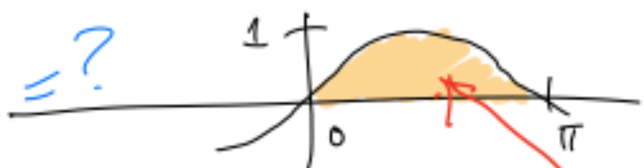
FTC $\int_a^b F'(x)\,dx = F(b) - F(a)$

$= F(x)\Big|_a^b$

F is called an antiderivative of $F'(x)$.

Examples

(1) $\int_0^{\pi} \sin(x)\,dx = ?$



Note $(\cos(x))' = -\sin(x)$, so

$(-\cos(x))' = \sin(x)$

$\Rightarrow \int_0^{\pi} \sin(x)\,dx = -\cos(x)\Big|_0^{\pi} = (-\underbrace{\cos(\pi)}_{-1}) - (-\underbrace{\cos(0)}_{1})$

$= -(-1) - -(1) = \boxed{2}$

② $\int_0^{\pi} \sin(x^2)\,dx = ?$

impossible to find an antiderivative of $\sin(x^2)$.

We could try $(-\cos(x^2))' = \sin(x^2)\cdot \underbrace{2x}_{\text{Not what we want.}}$

try $\left(\frac{-\cos(x^2)}{2x}\right)' = \frac{(2x)(\sin(x^2)\cdot 2x) - (-\cos(x^2))(2)}{4x^2}$

$= \ldots$ Also not what we want.

③ $\int_0^1 \frac{x^3 - 3x^2 + 2}{x^3}\,dx$

$= \int_0^1 \left(\frac{x^3}{x^3} - \frac{3x^2}{x^3} + \frac{2}{x^3}\right) dx$

$= \int_0^1 \left(1 - \frac{3}{x} + 2x^{-3}\right) dx$

antiderivative of $1 = x$

" of $\frac{-3}{x} = -3\cdot\frac{1}{x} = -3\ln x$

$(-3\ln x)' = -3\cdot\frac{1}{x} = \frac{-3}{x}$ ✓

antideriv of $2x^{-3} \to -x^{-2}$

$(x^{-2})' = -2x^{-3}$

$(-x^{-2})' = +2x^{-3}$

$= x - 3\ln x - x^{-2}\Big|_0^1 = (1 - 3\ln(1) - 1^{-2}) - (0 - 3\ln(0) - 0^{-2})$

$= \underline{\text{undefined}}$! !
(probably has infinite area!)
between 0 & 1.

(4) $\int_1^2 \frac{x^3 - 3x^2 + 2}{x^3}\,dx$

$$= x - 3\ln x - x^{-2} \Big|_1^2$$

$$= \left(2 - 3\ln(2) - 2^{-2}\right) - \left(\cancel{1} - 3\underbrace{\ln(1)}_{0} - \cancel{1^{-2}}\right)$$

$$= 2 - 3\ln(2) - \frac{1}{4}$$

$$= \boxed{\frac{7}{4} - 3\ln 2}$$

(5) Power rule for antiderivatives.

If $p \in \mathbb{R}$, $\left(\frac{x^{p+1}}{p+1}\right)' = x^p$ if $p \neq -1$

$$(\ln x)' = x^{-1} = \frac{1}{x}$$

Antiderivative symbol — where $F'(x) = f(x)$.

$$\int f(x)\,dx = F(x) + \underline{C}$$

constant can be added

This is the notation for all possible antiderivatives.

In the examples we did:

① $\int \sin(x)\,dx = \underbrace{-\cos(x) + C}_{\text{derivative of this is } \sin(x).}$

③ $\int \frac{x^3 - 3x^2 + 2}{x^3}\,dx \underset{\substack{\text{after}\\ \text{work}}}{=} \underbrace{x - 3\ln x - x^{-2} + C}_{\text{derivative of this } = \frac{x^3-3x^2+2}{x^3}.}$

⑤ Power Rule

$$\int x^p\,dx = \begin{cases} \frac{x^{p+1}}{p+1} + C & \text{if } p \neq -1 \\ \ln(x) + C & \text{if } p = -1. \end{cases}$$

Actually

$$(\ln(-x))' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}.$$

only works if x is negative

$$(\ln(x))' = \frac{1}{x}$$

only works if x is positive

Combine into one formula.

$$(\ln(|x|))' = \frac{1}{x}.$$

Beefier Power Rule for Antiderivatives:

$$\int x^p\,dx = \begin{cases} \dfrac{x^{p+1}}{p+1} + C & \text{if } p \neq -1 \\ \ln|x| + C & \text{if } p = -1. \end{cases}$$

6. $$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$$

since $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$,

$$= \arcsin(x)\Big|_0^{1/2} = \arcsin\left(\tfrac{1}{2}\right) - \arcsin(0)$$

$$= \frac{\pi}{6} - 0 = \boxed{\frac{\pi}{6}}.$$

7. $$\int_0^{\pi/9} \cos(3x)\,dx = ?$$

Since $(\sin(3x))' = \cos(3x)\cdot 3$,

$$\left(\frac{\sin(3x)}{3}\right)' = \frac{1}{3}\cdot(\sin(3x))' = \frac{1}{3}\cdot\cos(3x)\cdot 3 = \cos(3x)\ \checkmark$$

So $\int \cos(3x)\,dx = \frac{\sin(3x)}{3} + C$

$$\int_0^{\pi/9} \cos(3x)\,dx = \frac{\sin(3x)}{3}\Big|_0^{\pi/9}$$

$$= \frac{\sin\left(3\left(\frac{\pi}{9}\right)\right)}{3} - \frac{\sin(3(0))}{3}$$

$$= \frac{\sin\left(\frac{\pi}{3}\right)}{3} - \frac{\sin(0)}{3} = \frac{\sqrt{3}/2}{3} - 0$$

$$= \boxed{\frac{\sqrt{3}}{6}}$$

Words: $\int_a^b f(x)\,dx$ ← called a definite integral.

$\int f(x)\,dx$ ← called an indefinite integral (or general antiderivative)

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(8) $\int \frac{2x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\cdot(2x)\,dx$

$$\left[\ln(1+x^2)\right]' = \frac{1}{1+x^2}\cdot 2x$$

$$= \boxed{\ln(1+x^2) + C}$$

9 $$\int \frac{2x}{1+x^4}\,dx = \int \frac{1}{1+(x^2)^2}\cdot 2x\,dx$$

$$\left[\arctan(x^2)\right]' = \frac{1}{1+(x^2)^2}\cdot 2x$$

$$= \arctan(x^2) + C$$

10 $$\int_0^1 \frac{1+2x}{1+x^2}\,dx = \int_0^1 \left(\frac{1}{1+x^2} + \frac{2x}{1+x^2}\right) dx$$

$$= \arctan(x) + \ln(1+x^2) \Big|_0^1$$

$$= \left(\arctan(1) + \ln(1+1)\right) - \left(\arctan(0) + \ln(1+0)\right)$$

$$= \frac{\pi}{4} + \ln 2 - (0+0)$$

$$= \boxed{\frac{\pi}{4} + \ln 2}.$$

Techniques for finding complicated antiderivatives.

① $\int c \cdot f(x)\,dx = c\int f(x)\,dx$

② $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

③ Substitution – inverse of the chain rule.

$$\int \underbrace{f'(g(x))}_{f'(g)}\,\underbrace{g'(x)\,dx}_{dg} = f(g(x)) + C$$

"u substitution"

$$\int f'(u)\,du = f(u) + C$$

$$\int f'(u(x))\,u'(x)\,dx = f(u(x)) + C.$$

Idea: If your integral has an inside function – let that $= u$.
Then find $du = u'(x)\,dx$.
Substitute everything back into the integral so that u is the variable now → hopefully the integral is simpler.

(11) $\int e^{\cos(x)} \sin(x)\,dx = ?$

Let $u = \cos(x)$

$du = -\sin(x)\,dx$ $\quad \sin(x)\,dx = -du$

$= \int e^{u}(-du) = -\int e^{u}\,du$

$= -e^{u} + C$

$= \boxed{-e^{\cos(x)} + C}$

Check: $\left(-e^{\cos(x)}\right)' = -e^{\cos(x)} \cdot (\cos(x))'$

$= -e^{\cos(x)} \cdot (-\sin(x)) =$

$= e^{\cos(x)} \cdot \sin(x).$ ✓